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An algorithmic study on the integration of holonomic hyperfunctions — oscillatory integrals and a phase space integral associated with a Feynman diagram

By

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Abstract

Let $u(x, y)$ be a generalized function which satisfies a holonomic system M of linear differential equations with polynomial coefficients. Suppose that $u(x, y)$ is integrable with respect to x and let $v(y)$ be its integral. We give a sufficient condition for $v(y)$ to satisfy the D-module theoretic integration module of M, which can be computed algorithmically. We present some examples related to oscillatory integrals and Cutkosky-type phase space integrals associated with Feynman diagrams

§ 1. Introduction

In this paper, we call a distribution, or more generally, a hyperfunction *holonomic* for short, if it satisfies a holonomic system of linear differential equations with *polynomial* coefficients. The integration of a holonomic function with respect to some of its variables is again holonomic if the integrand is 'rapidly decreasing' with respect to the integration variables. Moreover, a holonomic system for the integral is defined naturally as a D-module and is computable, at least in theory, under this condition.

First we give a sufficient condition for the integral to be well-defined and to satisfy the D-module theoretic integration module, or the direct image. This allows us to treat, e.g., the oscillatory integral with a polynomial phase and a holonomic distribution as the amplitude function which is 'rapidly decreasing' with respect to the integration variables.

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Then we give some examples of computation of holonomic systems for such oscillatory integrals and their Fourier transforms, as well as what are called Cutkosky-type phase space integrals associated with Feynman diagrams.

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§ 2. Integration of generalized functions

Let $\varpi : \mathbb{R}^{n+d} \ni (x, y) \mapsto y \in \mathbb{R}^d$ be the projection with the standard coordinates $x = (x_1, \dots, x_n)$ of $\mathbb{R}^n$ and $y = (y_1, \dots, y_d)$ of $\mathbb{R}^d$.

Then the integration along the fibers of ϖ gives a sheaf homomorphism $\varpi_! \mathcal{B}_{\mathbb{R}^{n+d}} \to \mathcal{B}_{\mathbb{R}^d}$, and hence, in particular, a homomorphism

$$\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}}) \longrightarrow \Gamma(U, \mathcal{B}_{\mathbb{R}^d})$$

for an open set U of $\mathbb{R}^d$. Here $\mathcal{B}_{\mathbb{R}^n}$ stands for the sheaf of hyperfunctions on $\mathbb{R}^n$ and $\varpi_!$ the sheaf-theoretic direct image with proper supports.

For example, for a real polynomial f in x, and a distribution φ on $\mathbb{R}^n$, we are interested in the integrals

$$I(f, \varphi)(t) = \int_{\mathbb{R}^n} \delta(t - f(x))\varphi(x)\,dx, \qquad \hat{I}(f, \varphi)(t) = \int_{\mathbb{R}^n} e^{itf(x)}\varphi(x)\,dx.$$

$I(f, \varphi)$ and $\hat{I}(f, \varphi)$ are related by Fourier transformation. If φ is a probability density function, then $I(f, \varphi)$ is that of the random variable $f(x)$, and $\hat{I}(f, \varphi)$ is the characteristic function. We do not assume that φ has a compact support. Hence the integrands do not belong to $\Gamma(\mathbb{R}, \varpi_! \mathcal{B}_{\mathbb{R}^{n+1}})$ in general.

Definition 2.1. We call a pair of classes $(\mathcal{F}_{n,d}, \mathcal{F}_{0,d})$ adapted to the projection $\varpi : \mathbb{R}^{n+d} \to \mathbb{R}^d$ if the following conditions are satisfied:

1. $\mathcal{F}_{n,d}$ is a left module over the ring $D_{n+d} = \mathbb{C}\langle x, y, \partial_x, \partial_y\rangle$ of differential operators with polynomial coefficients in the variables (x, y) with the notation $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$, $\partial_{x_j} = \partial/\partial x_j$.

2. $\mathcal{F}_{0,d}$ is a left module over $D_d = \mathbb{C}\langle y, \partial_y\rangle$.

3. There exists a $\mathbb{C}$-linear map $\varpi_* : \mathcal{F}_{n,d} \longrightarrow \mathcal{F}_{0,d}$.

4. For any $u \in \mathcal{F}_{n,d}$, $P \in D_d$, and $j = 1, \dots, n$, one has

$$P\varpi_*(u) = \varpi_*(Pu), \qquad \varpi_*(\partial_{x_j} u) = 0.$$

The first example of a pair adapted to ϖ is $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$ and $\Gamma(U, \mathcal{B}_{\mathbb{R}^d})$ with $\varpi_*(u(x,y)) = \int_{\mathbb{R}^n} u(x,y)\,dy$ for $u \in \Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$.

As the second example, let $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ be the subspace of $\mathcal{S}'(\mathbb{R}^{n+d})$ consisting of distributions of the form

$$u(x,y) = \sum_{j=1}^{m} u_j(x) v_j(x,y) \qquad (m \in \mathbb{N},\ u_j \in \mathcal{S}(\mathbb{R}^n),\ v_j \in \mathcal{S}'(\mathbb{R}^{n+d})), \tag{2.1}$$

where $\mathcal{S}$ and $\mathcal{S}'$ denote the space of rapidly decreasing functions and that of tempered distributions respectively. Then $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ is a left D_{n+d}-submodule of $\mathcal{S}'(\mathbb{R}^{n+d})$.

As a special case $d = 0$, we denote by $\mathcal{SS}'(\mathbb{R}^n)$ the subspace of $\mathcal{S}'(\mathbb{R}^n)$ consisting of distributions of the form

$$u(x) = \sum_{j=1}^{m} u_j(x) v_j(x) \qquad (m \in \mathbb{N},\ u_j \in \mathcal{S}(\mathbb{R}^n),\ v_j \in \mathcal{S}'(\mathbb{R}^n)).$$

For a distribution $u(x,y)$ in $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$, the integral $\varpi_*(u(x,y)) = \int_{\mathbb{R}^n} u(x,y)\,dx$ is defined by the pairing

$$\left\langle \int_{\mathbb{R}^n} u(x,y)\,dx,\ \varphi(y) \right\rangle = \sum_{j=1}^{m} \langle v_j(x,y),\ u_j(x)\varphi(y) \rangle \qquad (\forall \varphi \in \mathcal{S}(\mathbb{R}^d)).$$

It does not depend on the choice of expression (2.1). In fact, assume $u(x,y) = 0$ in (2.1) and take $\chi(x)$ which belongs to the space $C_0^\infty(\mathbb{R}^n)$ of infinitely differentiable functions with compact support such that $\chi(x) = 1$ if $|x| \le 1$. Then for an arbitrary constant $r > 0$, we have an equality

$$0 = \left\langle \sum_{j=1}^{m} u_j(x) v_j(x,y),\ \chi(x/r)\varphi(y) \right\rangle = \sum_{j=1}^{m} \langle v_j(x,y),\ \chi(x/r) u_j(x) \varphi(y) \rangle$$

for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since $\chi(x/r)u_j(x)\varphi(y)$ converges to $u_j(x)\varphi(y)$ in $\mathcal{S}(\mathbb{R}^{n+d})$ as $r \to \infty$, we get

$$\sum_{j=1}^{m} \langle v_j(x,y),\ u_j(x)\varphi(y) \rangle = 0.$$

Proposition 2.2 (differentiation under the integral sign). *Suppose that $u(x,y)$ belongs to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$ with an open set U of $\mathbb{R}^d$, or else to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then*

$$P(y, \partial_y) \int_{\mathbb{R}^n} u(x,y)\,dx = \int_{\mathbb{R}^n} P(y, \partial_y) u(x,y)\,dx$$

holds for any $P = P(y, \partial_y) \in D_d$.

Proof. First we suppose $u(x,y)$ belongs to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$ and is defined by

$$u(x,y) = \sum_{j=1}^{m} u_j(x) v_j(x,y)$$

with $u_j \in \mathcal{S}(\mathbb{R}^n)$ and $v_j \in \mathcal{S}'(\mathbb{R}^{n+d})$. Then for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\left\langle \partial_{y_i} \int_{\mathbb{R}^n} u(x,y)\,dx,\ \varphi(y) \right\rangle = -\left\langle \int_{\mathbb{R}^n} u(x,y)\,dx,\ \partial_{y_i}\varphi(y) \right\rangle \\
&= -\sum_{j=1}^{m} \langle v_j(x,y),\ u_j(x)\partial_{y_i}\varphi(y)\rangle = -\sum_{j=1}^{m} \langle v_j(x,y),\ \partial_{y_i}(u_j(x)\varphi(y))\rangle \\
&= \sum_{j=1}^{m} \langle \partial_{y_i} v_j(x,y),\ u_j(x)\varphi(y)\rangle = \left\langle \int_{\mathbb{R}^n} \partial_{y_i} u(x,y)\,dx,\ \varphi(y) \right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle y_i \int_{\mathbb{R}^n} u(x,y)\,dx,\ \varphi(y) \right\rangle = \sum_{j=1}^{m} \langle v_j(x,y),\ y_i u_j(x)\varphi(y)\rangle \\
&= \sum_{j=1}^{m} \langle y_i v_j(x,y),\ u_j(x)\varphi(y)\rangle = \left\langle \int_{\mathbb{R}^n} y_i u(x,y)\,dx,\ \varphi(y) \right\rangle .
\end{aligned}$$

Next, let us assume $u(x,y)$ to belong to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$. Moreover, by induction on n, we may assume $n = 1$. Since the statement is local in U, we may suppose that U is convex and the support of $u(x,y)$ is contained in $[-R/2, R/2] \times U$ with $R > 0$. Then there exists a hyperfunction $v(x,y)$ on $\mathbb{R} \times U$ such that $\partial_x v(x,y) = u(x,y)$ whose singular spectrum S.S. $v(x,y)$ does not contain the points $(\pm R, y; \pm\sqrt{-1}dx)$ with $y \in U$ in the purely imaginary cotangent bundle $\sqrt{-1}T^*(\mathbb{R} \times \mathbb{R}^d)$. See Proposision 3.2.1 and subsequent arguments in Kashiwara-Kawai-Kimura [3] on integration of hyperfunctions. This implies that x is a real analytic parameter of $v(x,y)$ on a neighborhood of $\{\pm R\} \times U$. Hence $v(\pm R, y)$ are well-defined as hyperfunctions on U and one has

$$\int_{-\infty}^{\infty} u(x,y)\,dx = v(R,y) - v(-R,y)$$

by the definition. One also has

$$\int_{-\infty}^{\infty} P(y,\partial_y) u(x,y)\,dx = P(y,\partial_y) v(R,y) - P(y,\partial_y) v(-R,y)$$

for any $P \in D_d$ since $\partial_x P(y,\partial_y) v(x,y) = P(y,\partial_y) u(x,y)$. Thus we get

$$\begin{aligned}
P(y,\partial_y) \int_{-\infty}^{\infty} u(x,y)\,dx &= P(y,\partial_y) v(R,y) - P(y,\partial_y) v(-R,y) \\
&= \int_{-\infty}^{\infty} P(y,\partial_y) u(x,y)\,dx.
\end{aligned}$$

$\square$

Proposition 2.3. Suppose that $u(x,y)$ belongs to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$ with an open set U of $\mathbb{R}^d$, or else to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. Then one has

$$\int_{\mathbb{R}^n} \partial_{x_j} u(x,y)\, dx = 0 \qquad (j = 1, \dots, n).$$

Proof. First suppose that $u(x,y)$ belongs to $\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d)$. We may assume, without loss of generality, that $u(x,y) = v(x)w(x,y)$ with $v \in \mathcal{S}(\mathbb{R}^n)$ and $w \in \mathcal{S}'(\mathbb{R}^{n+d})$. Then for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}
&\left\langle \int_{\mathbb{R}^n} \partial_{x_j}(v(x)w(x,y))\, dx,\ \varphi(y) \right\rangle \\
&= \left\langle \int_{\mathbb{R}^n} (\partial_{x_j} v(x)) w(x,y)\, dx,\ \varphi(y) \right\rangle + \left\langle \int_{\mathbb{R}^n} v(x)(\partial_{x_j} w(x,y))\, dx,\ \varphi(y) \right\rangle \\
&= \langle w(x,y),\, (\partial_{x_j} v(x))\varphi(y) \rangle + \langle \partial_{x_j} w(x,y),\, v(x)\varphi(y) \rangle \\
&= \langle w(x,y),\, (\partial_{x_j} v(x))\varphi(y) \rangle - \langle w(x,y),\, \partial_{x_j}(v(x)\varphi(y)) \rangle = 0.
\end{aligned}$$

Next, let us assume that $u(x,y)$ belongs to $\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}})$. We may also assume that U is convex, $n = 1$, and the support of $u(x,y)$ is contained in $[-R/2, R/2] \times U$ with $R > 0$. Then by the definition of the integration, we have

$$\int_{-\infty}^{\infty} \partial_x u(x,y)\, dx = u(R,y) - u(-R,y) = 0.$$

$\square$

Hence the pairs $(\Gamma(U, \varpi_! \mathcal{B}_{\mathbb{R}^{n+d}}), \Gamma(U, \mathcal{B}_{\mathbb{R}^d}))$ and $(\mathcal{SS}'(\mathbb{R}^n \times \mathbb{R}^d),\ \mathcal{S}'(\mathbb{R}^d))$ are adapted to the projection ϖ of $\mathbb{R}^{n+d}$ to $\mathbb{R}^d$.

§ 3. Integration of D-modules

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_d)$ be (complex or real) variables. We set $X = \mathbb{C}^{n+d}$ and $Y = \mathbb{C}^d$ and let $\varpi_{\mathbb{C}} : X \ni (x,y) \longmapsto y \in Y$ be the projection. We denote by $D_X = D_{n+d}$ the ring of differential operators in the variables (x,y), and by $D_Y = D_d$ that in the variables y. The module

$$D_{Y \leftarrow X} := D_X / (\partial_{x_1} D_X + \cdots + \partial_{x_n} D_X)$$

has a structure of (D_Y, D_X)-bimodule. The *integral* of a left D_X-module M along the fibers of $\varpi_{\mathbb{C}}$, or the *direct image* by $\varpi_{\mathbb{C}}$ is defined to be the left D_Y-module

$$(\varpi_{\mathbb{C}})_* M := D_{Y \leftarrow X} \otimes_{D_X} M = M / (\partial_{x_1} M + \cdots + \partial_{x_n} M).$$

For an element u of M, let $[u]$ be its residue class in $(\varpi_{\mathbb{C}})_*M$. If M is generated by $u_1, \dots, u_r$ over D_X, then $(\varpi_{\mathbb{C}})_*M$ is generated by the set $\{x^\alpha [u_j] \mid 1 \le j \le r,\ \alpha \in \mathbb{N}^n\}$ over D_Y.

Let $(\mathcal{F}_{n,d}, \mathcal{F}_n)$ be a pair adapted to ϖ. Let h be a D_X-homomorphism from M to $\mathcal{F}_{n,d}$. Let us define a $\mathbb{C}$-linear map h' from M to $\mathcal{F}_d$ by

$$h'(u) = \varpi_*(h(u)) \qquad (\forall u \in M),$$

which is D_Y-linear and we have

$$\partial_{x_1} M + \cdots + \partial_{x_n} M \subset \ker h'.$$

Hence h' induces a D_Y-homomorphism

$$\varpi_*(h) : (\varpi_{\mathbb{C}})_* M \longrightarrow \mathcal{F}_d.$$

In conclusion, we have defined a $\mathbb{C}$-linear map

$$\varpi_* \ : \ \mathrm{Hom}_{D_X}(M, \mathcal{F}_{n,d}) \longrightarrow \mathrm{Hom}_{D_Y}((\varpi_{\mathbb{C}})_*M, \mathcal{F}_d).$$

If M is a holonomic D_X-module, then $(\varpi_{\mathbb{C}})_*M$ is a holonomic D_Y-module. An algorithm to compute $(\varpi_{\mathbb{C}})_*M$, which works at least if M is holonomic, was given in [9], [10]; see also [8]. For practical computation, we use a library file `nk_restriction.rr` by Hiromasa Nakayama for the computer algebra system Risa/Asir [7].

§ 4. Oscillatory integrals

Let $f(x)$ be a real polynomial in the real variables $x = (x_1, \dots, x_n)$. Suppose that $\varphi(x)$ belongs to $\mathcal{SS}'(\mathbb{R}^n)$. Let t and τ be real variables. Since both $\delta(t - f(x))\varphi(x)$ and $e^{itf(x)}\varphi(x)$ belong to $\mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}_t)$, the integrals

$$F(t) = I(f,\varphi)(t) = \int_{\mathbb{R}^n} \delta(t - f(x))\varphi(x)\,dx, \qquad G(t) = \hat{I}(f,\varphi)(t) = \int_{\mathbb{R}^n} e^{itf(x)}\varphi(x)\,dx$$

are well-defined as elements of $\mathcal{S}'(\mathbb{R})$. The integral $\hat{I}(f,\varpi)(t)$ is called the oscillatory integral with the phase function $f(x)$ and the amplitude function $\varphi(x)$, which is usually assumed to belong to $C_0^\infty(\mathbb{R}^n)$ in the literature (see e.g., [5]).

Proposition 4.1. *Define $F(t)$ and $G(\tau)$ as above with $\varphi \in \mathcal{SS}'(\mathbb{R}^n)$ and $f \in \mathbb{R}[x]$. Then $F(t)$ and $G(\tau)$ are related by*

$$G(\tau) = \hat{F}(\tau) := \int_{-\infty}^{\infty} e^{it\tau} F(t)\,dt, \qquad F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\tau} G(\tau)\,d\tau,$$

where the integrals make sense as Fourier transformation in $\mathcal{S}'(\mathbb{R})$. Moreover $G(\tau)$ belongs to $C^\infty(\mathbb{R})$.

Proof. We may assume that $\varphi(x) = \psi(x)u(x)$ with $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then by the definition of the integral of an element of $\mathcal{SS}'(\mathbb{R}^n_x \times \mathbb{R}_t)$, we get, for any $\chi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
\langle \hat{F}, \chi\rangle &= \left\langle \int_{\mathbb{R}^n} \psi(x)\delta(t - f(x))u(x)\,dx,\ \hat{\chi}\right\rangle = \langle \delta(t-f(x))u(x),\ \psi(x)\hat{\chi}(t)\rangle \\
&= \langle \delta(t)u(x),\ \psi(x)\hat{\chi}(t+f(x))\rangle = \langle u(x),\ \psi(x)\hat{\chi}(f(x))\rangle_x \\
&= \left\langle u(x),\ \psi(x)\int_{-\infty}^{\infty} e^{itf(x)}\chi(t)\,dt\right\rangle_x \\
&= \int_{-\infty}^{\infty}\left\langle u(x),\ \psi(x)e^{itf(x)}\chi(t)\right\rangle_x dt = \int_{-\infty}^{\infty}\left\langle u(x),\ \psi(x)e^{itf(x)}\right\rangle_x \chi(t)\,dt
\end{aligned}$$

in view of the lemma below. This implies

$$\hat{F}(\tau) = \left\langle u(x),\ \psi(x)e^{i\tau f(x)}\right\rangle_x = \int_{\mathbb{R}^n} \psi(x)e^{i\tau f(x)}u(x)\,dx = G(\tau)$$

and that $G(\tau)$ belongs to $C^\infty(\mathbb{R})$. □

Lemma 4.2. *Assume that $f(x)$ is a real polynomial in x, χ belongs to $\mathcal{S}(\mathbb{R})$, and that ψ belongs to $\mathcal{S}(\mathbb{R}^n)$. Then*

$$\psi(x)\int_{-\infty}^{\infty} e^{itf(x)}\chi(t)\,dt = \lim_{R\to\infty}\lim_{N\to\infty}\frac{R}{N}\sum_{k=-N}^{N-1}\psi(x)\exp\Big(i\frac{Rk}{N}f(x)\Big)\chi\Big(\frac{Rk}{N}\Big)$$

holds in the topology of $\mathcal{S}(\mathbb{R}^n)$.

Proof. Since it is easy to see that

$$\psi(x)\int_{-\infty}^{\infty} e^{itf(x)}\chi(t)\,dt = \psi(x)\lim_{R\to\infty}\int_{-R}^{R} e^{itf(x)}\chi(t)\,dt$$

holds in the topology of $\mathcal{S}(\mathbb{R}^n)$, let us show

$$\psi(x)\int_{-R}^{R} e^{itf(x)}\chi(t)\,dt = \lim_{N\to\infty}\frac{R}{N}\sum_{k=-N}^{N-1}\psi(x)\exp\Big(i\frac{Rk}{N}f(x)\Big)\chi\Big(\frac{Rk}{N}\Big)$$

in $\mathcal{S}(\mathbb{R}^n)$. For integers k, ν with $-N \le k \le N-1$ and $\nu \ge 0$, there exist t_k, t'_k in the interval $\left[\frac{R}{N}k,\, \frac{R}{N}(k+1)\right]$, which depend on x, such that

$$\int_{-R}^{R} e^{itf(x)}t^\nu\chi(t)\,dt = \frac{R}{N}\sum_{k=-N}^{N-1}\left\{\cos(t_kf(x))t_k^\nu\chi(t_k) + i\sin(t'_kf(x)){t'_k}^\nu\chi(t'_k)\right\}.$$

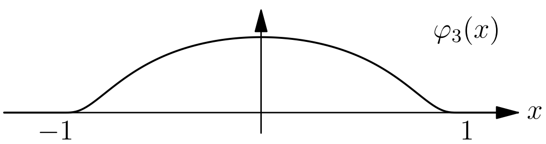
Hence

$$\begin{aligned}&\left|\int_{-R}^{R} e^{itf(x)}t^\nu\chi(t)\,dt-\frac{R}{N}\sum_{k=-N}^{N-1}\exp\Bigl(i\frac{Rk}{N}f(x)\Bigr)\Bigl(\frac{Rk}{N}\Bigr)^\nu\chi\Bigl(\frac{Rk}{N}\Bigr)\right|\\
&\le\frac{R}{N}\sum_{k=-N}^{N-1}\Bigl|\bigl\{\cos(t_kf(x))t_k^\nu\chi(t_k)+i\sin(t'_kf(x)){t'_k}^\nu\chi(t'_k)\bigr\}\\
&\qquad\qquad-\exp\Bigl(i\frac{Rk}{N}f(x)\Bigr)\Bigl(\frac{Rk}{N}\Bigr)^\nu\chi\Bigl(\frac{Rk}{N}\Bigr)\Bigr|\\
&\le\frac{R}{N}\sum_{k=-N}^{N-1}C(|f(x)|+1)\max\left\{\left|t_k-\frac{Rk}{N}\right|,\ \left|t'_k-\frac{Rk}{N}\right|\right\}\\
&\le\frac{2R^2}{N}C(|f(x)|+1)\end{aligned}$$

with some constant C independent of x. This implies the assertion. □

Example 4.3. Set $n=1$ and $x=x_1$. Let us choose

$$\varphi_1(x)=\exp\Bigl(-\frac{x^2}{2}\Bigr),\quad \varphi_2(x)=Y(1-x^2),\quad \varphi_3(x)=\begin{cases}\exp\Bigl(-\dfrac{1}{1-x^2}\Bigr) & \text{if } |x|<1,\\ 0 & \text{otherwise}\end{cases}$$



as amplitude functions and set

$$F_k(t)=\int_{-\infty}^{\infty}\delta(t-x^2)\varphi_k(x)\,dx,\quad G_k(\tau)=\int_{-\infty}^{\infty}e^{i\tau x^2}\varphi_k(x)\,dx\quad(k=1,2,3).$$

Here $Y(x)$ denotes the Heaviside function. Then $F_k(t)$ satisfy differential equations

$$(2t\partial_t+t+1)F_1(t)=0,\quad (t-1)(2t\partial_t+1)F_2(t)=0,\quad (2t(t-1)^2\partial_t+t^2+1)F_3(t)=0$$

respectively. The point 0 is a regular singular point of the three differential equations with characteristic exponent $-1/2$. The point 1 is a regular singular point of the second equation with characteristic exponent 0, but is an irregular singular point of the last equation. Consequently we get

$$F_1(t)=t_+^{-1/2}e^{-t/2},\qquad F_2(t)=t_+^{-1/2}Y(1-t)$$

in view of

$$\int_{-\infty}^{\infty}F_k(t)\,dt=\int_{-\infty}^{\infty}\varphi_k(x)\,dx,$$

and

$$F_3(t) = C_3 t_+^{-1/2} Y(1-t) \exp\Bigl(-\frac{1}{1-t}\Bigr)$$

with some constant C_3.

On the other hand, G_k $(k = 1, 2, 3)$ belong to $\mathcal{S}'(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and satisfy differential equations

$$\begin{aligned}
&((2\tau + i)\partial_\tau + 1)G_1(\tau) = 0,\\
&(2\tau\partial_\tau^2 + (-2i\tau + 3)\partial_\tau - i)G_2(\tau) = 0,\\
&(2\tau\partial_\tau^3 + (-4i\tau + 5)\partial_\tau^2 + (-2\tau - 8i)\partial_\tau - 1)G_3(\tau) = 0
\end{aligned}$$

respectively. In particular, we have $G_1(\tau) = \sqrt{2\pi}(1-2i\tau)^{-1/2}$. The equations for $G_2(\tau)$ and for $G_3(\tau)$ have 0 as a regular singular point and the point at infinity as an irregular singular point. Note that $G_2(\tau)$ and $G_3(\tau)$ are entire, i.e., holomorphic on $\mathbb{C}$.

Example 4.4. Set

$$F(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\Bigl(-\frac{|x|^2}{2}\Bigr) \delta(t - f(x))\, dx$$

with a quadratic form $f(x) = \sum_{i,j=1}^n a_{ij} x_i x_j$. If the absolute values of all the eigenvalues of the matrix (a_{ij}) are the same, then $F(t)$ satisfies a linear differential equation of the second order. We may assume

$$f(x) = a(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2)$$

with a constant $a > 0$. Then the integrand $u = u(x, t)$ satisfies

$$(t - f(x))u = (\partial_{x_i} + 2a x_i \partial_t + x_i)u = (\partial_{x_j} - 2a x_j \partial_t + x_j)u = 0 \qquad (1 \le i \le p < j \le n).$$

The following operators P and Q annihilate u:

$$\begin{aligned}
P &= \sum_{i=1}^p x_i(\partial_{x_i} + 2a x_i \partial_t + x_i) + \sum_{i=p+1}^n x_i(\partial_{x_i} - 2a x_i \partial_t + x_i)\\
&= \sum_{i=1}^n \partial_{x_i} x_i + 2f\partial_t + |x|^2 - n = \sum_{i=1}^n \partial_{x_i} x_i + 2\partial_t t + |x|^2 - n - 2\partial_t(t - f),\\
Q &= \sum_{i=1}^p x_i(\partial_{x_i} + 2a x_i \partial_t + x_i) - \sum_{i=p+1}^n x_i(\partial_{x_i} - 2a x_i \partial_t + x_i)\\
&= \sum_{i=1}^p \partial_{x_i} x_i - \sum_{i=p+1}^n \partial_{x_i} x_i + 2a|x|^2\partial_t + \frac{1}{a} f + n - 2p.
\end{aligned}$$

Hence

$$2a\partial_t P - Q = 2a\sum_{i=1}^{p} \partial_{x_i} x_i \partial_t - \sum_{i=1}^{p} \partial_{x_i} x_i + \sum_{i=p+1}^{n} \partial_{x_i} x_i + (-4a\partial_t^2 + \frac{1}{a})(t-f)$$
$$+ 4a\partial_t^2 t - 2na\partial_t - \frac{1}{a} t + (2p-n)$$

implies

$$\{4a^2 t\partial_t^2 + 2a^2(4-n)\partial_t - t + (2p-n)a\}F(t) = 0.$$

The solutions of this differential equation are expressed as

$$P\left\{\begin{array}{cccc} \multicolumn{2}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{\infty}} & 0 & \\ \dfrac{1}{2a} & 1-\dfrac{p}{2} & 0 & t \\ -\dfrac{1}{2a} & -\dfrac{1}{4}(2n-2p-4) & \dfrac{n-2}{2} & \end{array}\right\}.$$

On the other hand,

$$G(\tau) = \int_{\mathbb{R}^n} \exp\Big(i\tau f(x) - \frac{|x|^2}{2}\Big)$$

satisfies a differential equation

$$\{(4a^2\tau^2 + 1)\partial_\tau + a(2na\tau + (n-2p)i)\}G(\tau) = 0.$$

It follows that

$$G(\tau) = \exp\left(i\Big(p - \frac{n}{2}\Big)\tan^{-1}(2a\tau)\right)(4a^2\tau^2+1)^{-n/4}.$$

More generally, if $f(x)$ is a general quadratic form with eigenvalues $a_1, \dots, a_n$, then one has

$$G(\tau) = \prod_{k=1}^{n}(1-2ia_k\tau)^{-1/2} = \prod_{k=1}^{n}(4a_k^2\tau^2+1)^{-1/4}\exp\Big(-\frac{i}{2}\tan^{-1}(2a_k\tau)\Big)$$

since $G(\tau) = (1-2i\tau)^{-1/2}$ if $n = 1$ and $a = a_1 = 1$.

Example 4.5. Set $f(x,y) = x^3 - y^2$ and

$$F(t) = (2\pi)^{-1}\int_{\mathbb{R}^2} \exp\Big(-\frac{x^2+y^2}{2}\Big)\delta(t - f(x,y))\,dxdy.$$

Then $F(t)$ satisfies

$$\{108t^2\partial_t^5 + (-108t^2 + 648t)\partial_t^4 + (27t^2 - 486t + 627)\partial_t^3 + (85t - 303)\partial_t^2 \\ + (-4t + 21)\partial_t + t - 3\}F(t) = 0.$$

It has a regular singularity at 0 with the indicial equation $s(s-1)(s-2)(6s+1)(6s-7)$. Note that the b-function of f is $(s+1)(6s+5)(6s+7)$.

Example 4.6. Set $f(x, y, z) = x^2 - y^2 z$ and

$$F(t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp\Big(-\frac{x^2 + y^2 + z^2}{2}\Big)\delta(t - f(x, y, z))\, dxdydz.$$

Then $F(t)$ satisfies

$$\{16t^2\partial_t^5 + (16t^2 + 96t)\partial_t^4 + (4t^2 + 72t + 96)\partial_t^3 + (16t + 48)\partial_t^2 + (4t + 9)\partial_t + t + 3\}F(t) = 0.$$

It has a regular singularity at 0 with the indicial equation $s^2(s-1)^2(s-2)$. Note that the b-function of f is $(s+1)^2(2s+3)$.

Example 4.7. Set $f(x, y, z) = x^3 - y^2 z^2$ and

$$F(t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp\Big(-\frac{x^2 + y^2 + z^2}{2}\Big)\delta(t - f(x, y, z))\, dxdydz.$$

Then $F(t)$ satisfies a linear ordinary differential equation of order 10 which has a regular singularity at 0 with the indicial equation

$$s(s-1)(s-2)(s-3)(s-4)(s-5)(6s+1)^2(6s-7)^2 = 0.$$

Note that the b-function of f is $(s+1)(3s+4)(3s+5)(6s+5)^2(6s+7)^2$.

§ 5. Cutkosky-type phase space integrals associated with Feynman diagrams

The Cutkosky-type phase space integral associated with a Feynman diagram G, which we will call the phase space integral for short, describes the discontinuity of the Feynman integral $F_G(p)$ along its singularity locus. We consider simple Feynman diagrams in two-dimensional space-time for the sake of simplicity in actual computation, inspired by the recent work by Honda and Kawai (see e.g., [1], [2]) on the Landau-Nakanishi surface associated with G.

In general, for a two-dimensional vector $\mathbf{p} = (p_0, p_1)$, we denote $\mathbf{p}^2 = p_0^2 - p_1^2$ for the Lorentz norm and $d\mathbf{p} = dp_0 dp_1$ for the volume element. Let m be a positive constant.

Then the delta function $\delta(\mathbf{p}^2 - m^2)$ is well-defined and its support coincides with the curve $\mathbf{p}^2 - m^2 = 0$ in the 2-dimensional space-time $\mathbb{R}^2$. We set

$$\delta_+(\mathbf{p}^2 - m^2) = Y(p_0)\delta(\mathbf{p}^2 - m^2),$$

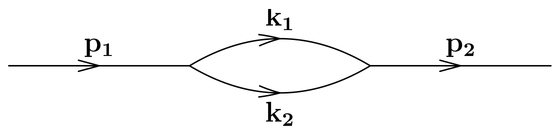
which is well-defined since the line $p_0 = 0$ is disjoint from the curve $\mathbf{p}^2 = m^2$. Its support is contained in $\{\mathbf{p} \mid \mathbf{p}^2 = m^2, p_0 \geq m\}$. Moreover, if $P \in D_2$ annihilates $\delta(\mathbf{p}^2 - m^2)$, then it also annihilates $\delta_+(\mathbf{p}^2 - m^2)$. More precisely it is easy to see that

$$\mathrm{Ann}_{D_2}\delta_+(\mathbf{p}^2 - m^2) = \mathrm{Ann}_{D_2}\delta(\mathbf{p}^2 - m^2)$$

holds, where D_2 is the ring of differential operators with polynomial coefficients with respect to the variables p_0, p_1. On the other hand, we denote by $\delta(\mathbf{p})$ the delta function $\delta(p_0)\delta(p_1)$ supported at the origin of $\mathbb{R}^2$.

We give a precise definition of the Cutkosky-type phase space integral associated with a Feynman diagram in each example instead of presenting a general formulation.

Example 5.1. Let us study the Feynman diagram G with two vertices, two external lines, and two internal lines as below:



Let us assign 2-vectors $\mathbf{p}_1 = (p_{10}, p_{11})$ and $\mathbf{p}_2 = (p_{20}, p_{21})$ to the external lines, $\mathbf{k}_1 = (k_{10}, k_{11})$ and $\mathbf{k}_2 = (k_{20}, k_{21})$ to the internal lines. Then the phase space integral associated with this diagram is defined to be

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \int \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2)\delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\, d\mathbf{k}_1 d\mathbf{k}_2.$$

It is easy to see that the product in the integrand makes sense as a hyperfunction by considering the singular spectrum (the analytic wave front set) of each factor. Integration with respect to $\mathbf{k}_2$ yields the expression

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2)I_G(\mathbf{p}_1)$$

with

$$I_G(\mathbf{p}_1) = \int \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2)\, d\mathbf{k}_1.$$

Since $I_G(\mathbf{p}_1)$ is invariant under Lorentz transformations of $\mathbf{p}_1$, we may put $\mathbf{p}_1 = (x, 0)$. Then the support of the integrand of $I_G((x, 0))$ is contained in the set

$$\begin{aligned} &\{(x, \mathbf{k}_1) \mid \mathbf{k}_1^2 - m_1^2 = (x - k_{10})^2 - k_{11}^2 - m_2^2 = 0,\ k_{10} > 0,\ x - k_{10} > 0\} \\ &\subset \{(x, \mathbf{k}_1) \mid k_{10} \geq m_1,\ x - k_{10} \geq m_2,\ |k_{11}| < k_{10}\} \\ &\subset \{(x, \mathbf{k}_1) \mid x \geq m_1 + m_2,\ m_1 \leq k_{10} \leq x - m_2,\ |k_{11}| < k_{10}\}. \end{aligned}$$

Hence the support of the integrand is proper with respect to the projection $\varpi : \mathbb{R}^3 \ni (x, \mathbf{k}_1) \mapsto x \in \mathbb{R}$. It follows that $I_G((x,0))$ is well-defined as a hyperfunction on $\mathbb{R}$ and its support is contained in $\{x \in \mathbb{R} \mid x \geq m_1 + m_2\}$.

Let us consider the second local cohomology group $H^2_I(\mathbb{C}[x, \mathbf{k}_1])$ with the ideal I generated by two polynomials $f_1 := \mathbf{k}_1^2 - m_1^2$ and $f_2 := (x - k_{10})^2 - k_{11}^2 - m_2^2$. Then we can identify the integrand with the cohomology class $\delta(f_1, f_2) = [1/(f_1 f_2)]$ in this local cohomology group. Since the variety $f_1 = f_2 = 0$ is non-singular, the annihilator in the ring D_3 of the integrand coincides with that of $\delta(f_1, f_2)$, which consists of first order operators together with f_1, f_2 and can be computed easily.

By virtue of Propositions 2.2 and 2.3, the integration algorithm described in [8] gives us a differential equation $PI_G((x,0)) = 0$ with

$$P = (x - m_1 - m_2)(x - m_1 + m_2)(x + m_1 - m_2)(x + m_1 + m_2)\partial_x + 2(x^2 - m_1^2 - m_2^2)x.$$

If $m_1 \neq m_2$, then we get

$$I_G((x,0)) = C(x - m_1 + m_2)^{-1/2}(x + m_1 - m_2)^{-1/2}(x + m_1 + m_2)^{-1/2}(x - m_1 - m_2)_+^{-1/2}$$

with some constant C by quadrature noting that its support is contained in the interval $[m_1 + m_2, \infty)$ and that there is no hyperfunction solution of the differential equation above whose support is the point $m_1 + m_2$.

On the other hand, if $m_1 = m_2$, then $I_G((x,0))$ satisfies

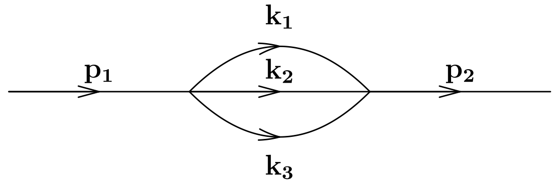
$$\{x(x^2 - 4m_1^2)\partial_x + 2(x^2 - 2m_1^2)\} I_G((x,0)) = 0.$$

It follows that

$$I_G((x,0)) = Cx^{-1}(x + 2m_1)^{-1/2}(x - 2m_1)_+^{-1/2}$$

with some constant C.

Example 5.2. The phase space integral associated with the Feynman diagram G with two vertices, two external lines, and three internal lines as below



is given by

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \int \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \\ \times \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\delta_+(\mathbf{k}_3^2 - m_3^2)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

with variables $\mathbf{p}_1 = (p_{10}, p_{11})$, $\mathbf{p}_2 = (p_{20}, p_{21})$, $\mathbf{k}_1 = (k_{10}, k_{11})$, $\mathbf{k}_2 = (k_{20}, k_{21})$, $\mathbf{k}_3 = (k_{30}, k_{31})$ and positive constants m_1, m_2, m_3. We rewrite this integral as

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \delta(\mathbf{p}_1 - \mathbf{p}_2) I_G(\mathbf{p}_1)$$

with

$$I_G(\mathbf{p}_1) = \int \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\delta_+((\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)^2 - m_3^2)\, d\mathbf{k}_1 d\mathbf{k}_2.$$

As in the previous example we can set $\mathbf{p}_1 = (x, 0)$. The support of the integrand is contained in

$$\{(x, \mathbf{k}_1, \mathbf{k}_2) \mid k_{10} \geq m_1,\ k_{20} \geq m_2,\ x - k_{10} - k_{20} \geq m_3,\ |k_{11}| < k_{10},\ |k_{21}| < k_{20}\}.$$

Hence $I_G((x, 0))$ is well-defined as a hyperfunction on $\mathbb{R}$ and its support is contained in the interval $[m_1 + m_2 + m_3, \infty)$.

Since the computation for general m_1, m_2, m_3 is intractable, let us set $m_1 = m_2 = m_3 = 1$. Then by the integration algorithm we obtain

$$\{(x(x-1)(x+1)(x-3)(x+3)\partial_x^2 + (5x^4 - 30x^2 + 9)\partial_x + 4x^3 - 12x\} I_G((x, 0)) = 0.$$

The points $0, \pm 1, \pm 3$ are regular singular and the indicial equations at these points are all s^2. It follows that $I_G((x, 0))$ is a locally integrable function of the form

$$I_G((x, 0)) = a(x) Y(x - 3)$$

with a real analytic function $a(x)$ on the interval $(1, \infty)$ in view of the lemma below and analytic continuation. Here we have $a(3) \neq 0$ unless $I_G((x, 0))$ vanishes everywhere. The point at infinity is also a regular singular point with the indicial equation $(s-2)^2$.

On the other hand, if we set $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, then we obtain

$$\begin{aligned}
&\{56x^2(x-2)(x+2)(x-4)(x+4)(x-6)(x+6)\partial_x^3 \\
&+ (-15x^9 + 1680x^7 - 46256x^5 + 341888x^3 - 387072x)\partial_x^2 \\
&+ (-75x^8 + 5544x^6 - 98000x^4 + 404480x^2 - 96768)\partial_x \\
&- 60x^7 + 3192x^5 - 33712x^3 + 44608x\} I_G((x, 0)) = 0.
\end{aligned}$$

The points $0, \pm 2, \pm 4, \pm 6$ are regular singular points. The indicial equation at 6 is $s^2(s-1)$. It follows that $I_G((x, 0))$ is a locally integrable function of the form

$$I_G((x, 0)) = a(x) Y(x - 6)$$

with a real analytic function $a(x)$ on the interval $(4, \infty)$ such that $a(6) \neq 0$ unless $I_G((x, 0))$ vanishes everywhere, in view of the lemma below. The point at infinity is an irregular singular point.

Lemma 5.3. *Let P be a differential operator of the form*

$$P = x\partial_x^m + a_1(x)\partial_x^{m-1} + \cdots + a_m(x)$$

with a positive integer m and analytic functions $a_1(x), \ldots, a_m(x)$ defined on a neighborhood of $x = 0$. Assume $a_1(0) = m - 1$. Let $u(x)$ be a hyperfunction defined on a neighborhood of 0 whose support is contatined in $\{x \in \mathbb{R} \mid x \geq 0\}$ such that $Pu(x) = 0$. Then $u(x)$ is written in the form

$$u(x) = a(x)Y(x)$$

with a real anayltic function $a(x)$ on a neighborhood of 0 such that $Pa(x) = 0$. Moreover, we have $a(0) \neq 0$ unless $u(x)$ vanishes everywhere.

Proof. Note that 0 is a regular singular point of P and its indicial polynomial at 0 is given by

$$\begin{aligned} b(\lambda) &= \lambda(\lambda-1)\cdots(\lambda-m+1) + a_1(0)\lambda(\lambda-1)\cdots(\lambda-m+2) \\ &= \lambda(\lambda-1)\cdots(\lambda-m+2)(\lambda-m+1+a_1(0)) \\ &= \lambda^2(\lambda-1)\cdots(\lambda-m+2). \end{aligned}$$

Since $b(\lambda) = 0$ has no integer roots greater than $m-2$, it follows that the homomorphism $P : \mathbb{C}\{x\} \to \mathbb{C}\{x\}$ is surjective and the dimension of its kernel is $m - 1$.

By the assumption, there exists a holomorphic function $F(z)$ on the set $\{z \in \mathbb{C} \mid |z| < \varepsilon\} \setminus [0, \infty]$ with $\varepsilon > 0$ such that

$$u(x) = F(x + \sqrt{-1}0) - F(x - \sqrt{-1}0).$$

Since $P : \mathbb{C}\{x\} \to \mathbb{C}\{x\}$ is surjective, we may assume $PF(z) = 0$. Let us rewrite P as

$$P = z\partial_z^m + (m-1)\partial_z^{m-1} + P_1\partial_z^{m-2} + \cdots + P_{m-2}\partial_z + P_{m-1} + P_m + \cdots,$$

where P_k is a differential operator of order at most $\min\{k, m-1\}$ such that

$$P_k z^\lambda = p_k(\lambda) z^{\lambda + \max\{0, k-m+1\}}$$

with a polynomial $p_k(\lambda)$ of λ. Following the Frobenius method, we can construct a series

$$v(z, \lambda) = \sum_{n=0}^{\infty} c_n(\lambda) z^{\lambda+n}$$

with rational functions $c_n(\lambda)$ of λ such that

$$Pv(z, \lambda) = b(\lambda) z^\lambda \tag{5.1}$$

and $c_0(\lambda) = 1$ by the recursion formula

$$
\begin{aligned}
c_n(\lambda) = &- \sum_{k=1}^{\max\{m-2,n\}} \frac{(\lambda+n-k)\cdots(\lambda+n-m+2)}{b(\lambda+n)} p_k(\lambda+n-m+1) c_{n-k}(\lambda) \\
&- \sum_{k=m-1}^{n} \frac{p_k(\lambda+n-k)}{b(\lambda+n)} c_{n-k}(\lambda) \\
= &- \sum_{k=1}^{\max\{m-2,n\}} \frac{p_k(\lambda+n-m+1)}{(\lambda+n)^2(\lambda+n-1)\cdots(\lambda+n-k+1)} c_{n-k}(\lambda) \\
&- \sum_{k=m-1}^{n} \frac{p_k(\lambda+n-k)}{b(\lambda+n)} c_{n-k}(\lambda)
\end{aligned}
$$

for $n = 1, 2, 3, \ldots$. This implies that $c_n(\lambda)$ are regular at $\lambda = 0$. Differentiating (5.1) with respect to λ and substituting 0 for λ, we get

$$
P\frac{\partial v}{\partial \lambda}(z,0) = \frac{\partial b}{\partial \lambda}(0) + b(0)\log z = 0
$$

with

$$
\frac{\partial v}{\partial \lambda}(z,0) = \sum_{n=0}^{\infty} \frac{\partial c_n}{\partial \lambda}(0) z^n + \sum_{n=0}^{\infty} c_n(0) z^n \log z.
$$

Hence $F(z)$ is written in the form

$$
F(z) = G(z) + a\sum_{n=0}^{\infty} c_n(0) z^n \log z
$$

with a holomorphic function $G(z)$ on a neighborhood of 0 and $a \in \mathbb{C}$. Hence we get

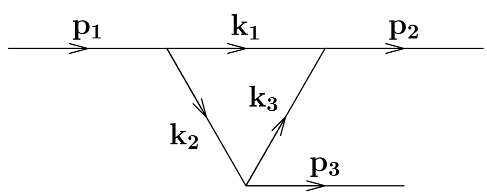
$$
u(x) = 2\pi\sqrt{-1}a\sum_{n=0}^{\infty} c_n(0) x^n Y(x) = 2\pi\sqrt{-1}av(x,0)Y(x)
$$

with $c_0(0) = 1$ and $Pv(x,0) = b(0) = 0$. □

Example 5.4. Let us consider the phase space integral

$$
\begin{aligned}
\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2) = \int & \delta(\mathbf{p}_1 - \mathbf{k}_1 - \mathbf{k}_2)\delta(-\mathbf{p}_2 + \mathbf{k}_1 + \mathbf{k}_3)\delta(-\mathbf{p}_3 + \mathbf{k}_2 - \mathbf{k}_3) \\
& \times \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+(\mathbf{k}_2^2 - m_2^2)\delta_+(\mathbf{k}_3^2 - m_3^2)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3
\end{aligned}
$$

associated with the diagram G below.



Performing the integration with respect to $\mathbf{k}_2$ and $\mathbf{k}_3$, we obtain

$$\tilde{I}_G(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) I_G(\mathbf{p}_1, \mathbf{p}_2)$$

with

$$I_G(\mathbf{p}_1, \mathbf{p}_2) = \int \delta_+(\mathbf{k}_1^2 - m_1^2)\delta_+((\mathbf{p}_1 - \mathbf{k}_1)^2 - m_2^2)\delta_+((\mathbf{p}_2 - \mathbf{k}_1)^2 - m_3^2)\, d\mathbf{k}_1.$$

The support of the integrand is contained in

$$\{(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_1) \mid k_{10} \geq m_1,\, p_{10} - k_{10} \geq m_2,\, p_{20} - k_{10} \geq m_3,\ |k_{11}| < k_{10}\}.$$

Hence $I_G(\mathbf{p}_1, \mathbf{p}_2)$ is well-defined as a hyperfunction on $\mathbb{R}^4$ and one has

$$\operatorname{supp} I_G(\mathbf{p}_1, \mathbf{p}_2) \subset \{(\mathbf{p}_1, \mathbf{p}_2) \mid p_{10} \geq m_1 + m_2,\ p_{20} \geq m_1 + m_3\}.$$

Let us set $m_1 = m_2 = m_3 = 1$, $\mathbf{p}_1 = (x, 0)$ and $\mathbf{p}_2 = ((y+z)/2, (y-z)/2)$ following Honda-Kawai-Stapp [2] and set

$$I_G(x, y, z) = I_G((x, 0), ((y+z)/2, (y-z)/2))$$

by abuse of notation. Then the integration algorithm returns a holonomic system $M = D_3/I$ for $I_G(x, y, z)$ with a left ideal I of D_3, which is too complicated to show here. The characteristic variety of M is given by

$$\begin{aligned}\mathrm{Char}(M) = {} & T^*_{\{f=0\}}\mathbb{C}^3 \cup T^*_{\{x=0\}}\mathbb{C}^3 \cup T^*_{\{x=f_0=0\}}\mathbb{C}^3 \cup T^*_{\{x=yz-4=0\}}\mathbb{C}^3 \\ & \cup T^*_{\{x=y=0\}}\mathbb{C}^3 \cup T^*_{\{x=z=0\}}\mathbb{C}^3 \cup T^*_{\{x=y=z=0\}}\mathbb{C}^3\end{aligned}$$

with

$$f(x, y, z) = yzx^2 - yz(y+z)x + y^2z^2 + (y-z)^2, \qquad f_0(y, z) = f(0, y, z),$$

where we denote by $T^*_Z\mathbb{C}^3$ the closure of the conormal bundle of the regular part of an analytic set Z of $\mathbb{C}^3$. The decomposition of $\mathrm{Char}(M)$ was done by using a library file `noro_pd.rr` of Risa/Asir [7] for prime and primary decomposition of polynomial ideals developed by M. Noro (see e.g., [4] for algorithms); he also computed a primary decomposition of the symbol ideal of I, which enabled us to compute the multiplicity of each component of $\mathrm{Char}(M)$. Thus the characteristic cycle, i.e., the characteristic variety with multiplicity of each component, of M is

$$\begin{aligned} & T^*_{\{f=0\}}\mathbb{C}^3 + 2\,T^*_{\{x=0\}}\mathbb{C}^3 + T^*_{\{x=f_0=0\}}\mathbb{C}^3 + T^*_{\{x=yz-4=0\}}\mathbb{C}^3 \\ & + T^*_{\{x=y=0\}}\mathbb{C}^3 + T^*_{\{x=z=0\}}\mathbb{C}^3 + 2\,T^*_{\{x=y=z=0\}}\mathbb{C}^3.\end{aligned}$$

In particular, the support of M as D-module is the hypersuface of $\mathbb{C}^3$ defined by $xf(x,y,z) = 0$. The singular locus of the complex hypersurface $f = 0$ is the union of two complex lines $\{x = y = z\}$ and $\{y = z = 0\}$. There is a stratification of the hypersurface $f = 0$ of $\mathbb{C}^3$ with respect to the (local) b-function $b_{f,p}(s)$ of f at a point p of each stratum as follows:

strata	$b_{f,p}(s)$
$\{(0,0,0)\}$	$(s+1)^3(2s+3)$
$\{(2,0,0),(-2,0,0),(2,2,2),(-2,-2,-2)\}$	$(s+1)^2(2s+3)$
$\{x=y=z\} \cup \{y=z=0\} \setminus \{(0,0,0),(\pm 2,0,0),\pm(2,2,2)\}$	$(s+1)^2$
$\{f=0\} \setminus (\{x=y=z\} \cup \{y=z=0\})$	$s+1$

Note that the b-function of f at the points $(\pm 2,0,0)$ and $\pm(2,2,2)$ coincides with that of $g := x^2 - y^2 z$ at the origin, which defines what is called the Whitney umbrella. More precisely, the b-function of g at each stratum is as follows:

stratum	$b_{g,p}(s)$
$\{(0,0,0)\}$	$(s+1)^2(2s+3)$
$\{x=y=0\} \setminus \{(0,0,0)\}$	$(s+1)^2$
$\{g=0\} \setminus \{x=y=0\}$	$s+1$

However, the present author does not know whether the germ of analytic function $(f,(2,2,2))$, for example, is (real) analytically equivalent to $(g,(0,0,0))$. We used a library file `nn_ndbf.rr` of Risa/Asir [7] for the computation of the stratifications above (see e.g., [6] for algorithms).

The fiber of the characteristic variety of M at each zero-dimensional stratum of $f = 0$ is given by

$$\pi^{-1}((0,0,0)) \cap \mathrm{Char}(M) = \{(0,0,0;\xi,\eta,\zeta) \mid \xi,\eta,\zeta \in \mathbb{C}\},$$
$$\pi^{-1}(p) \cap \mathrm{Char}(M) = \{(p;\xi,\eta,\zeta) \mid \xi,\eta,\zeta \in \mathbb{C},\ \eta = \zeta\}$$

with $p = (\pm 2,0,0)$ or $p = \pm(2,2,2)$, where $\pi : T^*\mathbb{C}^3 \to \mathbb{C}^3$ is the projection of the cotangent bundle to the base space. The fiber of $\mathrm{Char}(M)$ at a point in a one-dimensional stratum, e.g., $x = y = z$, consists of two complex lines while that at a regular point of $f = 0$ consists of one complex line.

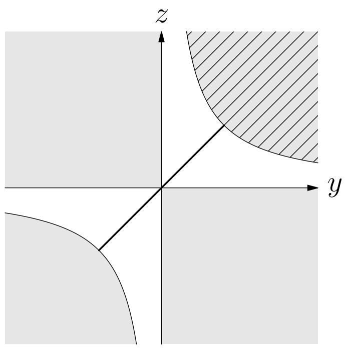
Next let us consider the real hypersurface

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$$

of $\mathbb{R}^3$. Since the discriminant of f with respect to x is $yz(y-z)^2(yz-4)$, the projection of S to the yz-plane is given by

$$S_{yz} = \{(y,z) \in \mathbb{R}^2 \mid yz(yz-4) \geq 0\} \cup \{(y,z) \mid y = z\}$$

as shown by the grey regions and the line segment in the figure below:



Except at the origin, the fiber of the projection of S to S_{yz} consists of one or two points. In particular, the fiber at a point in the line $y = z$ consists of only one point. Hence S has a line segment connecting the two points $\pm(2, 2, 2)$ as a one-dimensional component.

Since the support of $I_G(x, y, z)$ is contained in the support of M, i.e., $xf(x, y, z) = 0$, as well as in the set $\{(x, y, z) \in \mathbb{R}^3 \mid x \geq 2,\ y + z \geq 4\}$, we have

$$\operatorname{supp} I_G(x, y, z) \subset \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0,\ x \geq 2,\ y + z \geq 4\}.$$

In addition, since $I_G(x, y, z)$ coincides with a finite sum of derivatives of $\delta(f)$ multiplied by real analytic functions on the regular part S_{reg} of S, the intersection of the support of $I_G(x, y, z)$ and S_{reg} is a union of connected components of S_{reg}. Thus we conclude that the support of $I_G(x, y, z)$ satisfies

$$\operatorname{supp} I_G(x, y, z) \subset \{(x, y, z) \mid f(x, y, z) = 0,\ x \geq 2,\ y > 0,\ z > 0,\ yz \geq 4\},$$

the projection of which to the yz-plane is shown as the hatched region in the figure above. More precisely, we can confirm by direct computation that $xf(x, y, z)$ belongs to the left ideal I, which implies $f(x, y, z)I_G(x, y, z) = 0$. It follows that $I_G(x, y, z)$ is the product of a real analytic function and $\delta(f)$ on S_{reg}.

We remark that the hypersurface S of $\mathbb{R}^3$ coincides with the Landau-Nakanishi surface associated with the triangle diagram T_1 studied by Honda-Kawai-Stapp in Appendix A of [2].

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